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ON THE EXISTENCE OF MILD SOLUTIONS TO SOME SEMILINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the existence of a mild solution for some fractional semilinear differential equations with non local conditions. Using a more appropriate definition of a mild solution than the one given in [12], we prove the existence and uniqueness of such solutions, assuming that the linear part is the infinitesimal generator of an analytic semigroup that is compact for $t > 0$ and the nonlinear part is a Lipschitz continuous function with respect to the norm of a certain interpolation space. An example is provided.

1. Introduction

Let $\mathbb{X}$ be a Banach space and let $T > 0$. This paper is aimed at discussing about the existence and the uniqueness of a mild solution for the fractional semilinear integro-differential equation with nonlocal conditions in the form:

$$(1)\begin{cases} D^{\beta}\, x(t) = -Ax(t) + f(t, x(t)) + \displaystyle\int_0^t a(t-s)h(s, x(s))\, ds, & t \in [0, T], \\ x(0) + g(x) = x_0, \end{cases}$$

where the fractional derivative D^{β} $(0 < \beta < 1)$ is understood in the Caputo sense, the linear operator $-A$ is the infinitesimal generator of an analytic semigroup $(R(t))_{t\geq 0}$ that is uniformly bounded on $\mathbb{X}$ and compact for $t > 0$, the function $a(\cdot)$ is real-valued such that

$$a_T = \int_0^T a(s)\, ds < \infty, \tag{2}$$

the functions f, g and h are continuous, and the non local condition

$$g(x) = \sum_{k=1}^{p} c_k x(t_k),$$

with $c_k,\ k = 1, 2, ...p$, are given constants and $0 < t_1 < t_2 < ... < t_p \leq T$.

Let us recall that those nonlocal conditions were first utilized by K. Deng [4]. In his paper, K. Deng indicated that using the nonlocal condition $x(0) + g(x) = x_0$

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to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy Problem $x(0) = x_0$. Let us observe also that since Deng's paper, such problem has attracted several authors including A. Aizicovici, L. Byszewski, K. Ezzinbi, Z. Fan, J. Liu, J. Liang, Y. Lin, T.-J. Xiao, H. Lee, etc. (see for instance [1, 2, 3, 4, 9, 8, 7, 14, 11, 13] and the references therein).

This problem has been studied in Mophou and N'Guérékata [12]. In this paper, we revisit that work and use a more appropriate definition for mild solutions. Namely, we investigate the existence and the uniqueness of a mild solution for the fractional semilinear differential equation (1), assuming that f is defined on $[0,T] \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha$ where $\mathbb{X}_\alpha = D(A^\alpha)$ $(0 < \alpha < 1)$, the domain of the fractional powers of A.

The rest of this paper is organized as follows. In Section 2 we give some known preliminary results on the fractional powers of the generator of an analytic compact semigroup. In Section 3, we study the existence and the uniqueness of a mild solution for the fractional semilinear differential equation (1). We give an example to illustrate our abstract results.

2. Preliminaries

Let $I = [0,T]$ for $T > 0$ and let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. Let $\big(\mathbb{B}(\mathbb{X}), \|\cdot\|_{\mathbb{B}(\mathbb{X})}\big)$ be the Banach space of all linear bounded operators on $\mathbb{X}$ and $A : D(A) \to \mathbb{X}$ be a linear operator such that $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $(R(t))_{t\geq 0}$, which is compact for $t > 0$. In particular, this means that there exists $M > 1$ such that

$$\sup_{t\geq 0} \|R(t)\|_{\mathbb{B}(\mathbb{X})} \leq M. \tag{3}$$

Moreover, we assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$(see [8]). We have the following basic properties for fractional powers A^α of A:

Theorem 2.1. *([15], pp. 69 -75). Under previous assumptions, then:*

(i) $\mathbb{X}_\alpha = D(A^\alpha)$ *is a Banach space with the norm* $\|x\|_\alpha := \|A^\alpha x\|$ *for* $x \in D(A^\alpha)$*;*

(ii) $R(t) : \mathbb{X} \to \mathbb{X}_\alpha$ *for each* $t > 0$*;*

(iii) $A^\alpha R(t)x = R(t)A^\alpha x$ *for each* $x \in D(A^\alpha)$ *and* $t \geq 0$*;*

(iv) For every $t > 0$, $A^\alpha R(t)$ is bounded on $\mathbb{X}$ and there exist $M_\alpha > 0$ and $\delta > 0$ such that

$$\|A^\alpha R(t)\|_{\mathbb{B}(\mathbb{X})} \leq \frac{M_\alpha}{t^\alpha} e^{-\delta t}; \tag{4}$$

(v) $A^{-\alpha}$ is a bounded linear operator in $\mathbb{X}$ with $D(A^\alpha) = Im(A^{-\alpha})$; and

(vi) If $0 < \alpha \leq \nu$, then $D(A^\nu) \hookrightarrow D(A^\alpha)$.

Remark 2.2. Observe as in [9] that by Theorem 2.1 (ii) and (iii), the restriction $R_\alpha(t)$ of $R(t)$ to $\mathbb{X}_\alpha$ is exactly the part of $R(t)$ in $\mathbb{X}_\alpha$.
Let $x \in \mathbb{X}_\alpha$. Since

$$\|R(t)x\|_\alpha = \|A^\alpha R(t)x\| = \|R(t)A^\alpha x\| \leq \|R(t)\|_{\mathbb{B}(\mathbb{X})}\|A^\alpha x\| = \|R(t)\|_{\mathbb{B}(\mathbb{X})}\|x\|_\alpha,$$

and as t decreases to 0

$$\|R(t)x - x\|_\alpha = \|A^\alpha R(t)x - A^\alpha x\| = \|R(t)A^\alpha x - A^\alpha x\| \to 0,$$

for all $x \in \mathbb{X}_\alpha$, it follows that $(R(t))_{t\geq 0}$ is a family of strongly continuous semigroup on $\mathbb{X}_\alpha$ and $\|R_\alpha(t)\|_{\mathbb{B}(\mathbb{X})} \leq \|R(t)\|_{\mathbb{B}(\mathbb{X})}$ for all $t \geq 0$.

Lemma 2.3. [9] *The restriction $R_\alpha(t)$ of $R(t)$ to $\mathbb{X}_\alpha$ is an immediately compact semigroup in $\mathbb{X}_\alpha$, and hence it is immediately norm-continuous.*

Now, let Φ_β be the Mainardi function:

$$\Phi_\beta(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}.$$

Then

$$\Phi_\beta(t) \geq 0 \text{ for all } t > 0; \tag{5a}$$

$$\int_0^\infty \Phi_\beta(t)dt = 1; \tag{5b}$$

$$\int_0^\infty t^\eta \Phi_\beta(t)dt = \frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}, \quad \forall \eta \in [0,1]. \tag{5c}$$

For more details we refer to [10].
We set

$$\mathbb{S}_\beta(t) = \int_0^\infty \Phi_\beta(\theta)R(\theta t^\beta)\, d\theta, \tag{6}$$

$$\mathbb{P}_\beta(t) = \int_0^\infty \beta\theta\Phi_\beta(\theta)R(t^\beta\theta)d\theta \tag{7}$$

Then we have the following results

Lemma 2.4. [16] *Let $\mathbb{S}_\beta$ and $\mathbb{P}_\beta$ be the operators defined respectively by (6) and (7). Then*

(i) $\|\mathbb{S}_\beta(t)x\| \leq M\|x\|$; $\|\mathbb{P}_\beta(t)x\| \leq M\dfrac{\beta}{\Gamma(\beta+1)}\|x\|$ *for all* $x \in \mathbb{X}$ *and* $t \geq 0$.

(ii) The operators $(\mathbb{S}_\beta(t))_{t\geq 0}$ *and* $(\mathbb{P}_\beta(t))_{t\geq 0}$ *are strongly continuous.*

(iii) The operators $(\mathbb{S}_\beta(t))_{t>0}$ *and* $(\mathbb{P}_\beta(t))_{t>0}$ *are compact.*

Lemma 2.5. *Let $\mathbb{S}_\beta$ and $\mathbb{P}_\beta$ be the operators defined respectively by (6) and (7). Then*

$$\|\mathbb{S}_\beta(t)x\|_\alpha \leq M\|x\|_\alpha, \ \forall x \in \mathbb{X}_\alpha, \ t \geq 0,$$

$$\|\mathbb{P}_\beta(t)x\|_\alpha \leq \begin{cases} \dfrac{\beta M_\alpha t^{-\beta\alpha}\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\|x\| & \text{if} \quad x \in \mathbb{X}, \ t > 0, \\ \dfrac{\beta}{\Gamma(1+\beta)}\|x\|_\alpha & \text{if} \quad x \in \mathbb{X}_\alpha, \ t > 0. \end{cases}$$

Proof. Using (3) and (5b) we have for any $x \in \mathbb{X}_\alpha$ and $t \geq 0$,

$$\begin{aligned} \|\mathbb{S}_\beta(t)x\|_\alpha &= \left\| \int_0^\infty \Phi_\beta(\theta)R(\theta t^\beta)x\, d\theta x \right\|_\alpha \\ &\leq \int_0^\infty \Phi_\beta(\theta)\left\|A^\alpha R(\theta t^\beta)x\right\| d\theta x \\ &\leq M\int_0^\infty \Phi_\beta(\theta)\|A^\alpha x\|\, d\theta \\ &= M\|x\|_\alpha, \ \forall x \in \mathbb{X}_\alpha. \end{aligned}$$

In view of (4) and (5c), we can write for any $t > 0$,

$$\begin{aligned} \|\mathbb{P}_\beta(t)x\|_\alpha &= \left\| \int_0^\infty \beta\theta\Phi_\beta(\theta)R(\theta t^\beta)x\, d\theta \right\|_\alpha \\ &\leq \int_0^\infty \beta\theta\Phi_\beta(\theta)\left\|A^\alpha R(\theta t^\beta)x\right\| d\theta \\ &\leq \int_0^\infty \beta\theta\Phi_\beta(\theta)\|A^\alpha R(\theta t^\beta)\|_{\mathbb{B}(\mathbb{X})}\|x\|\, d\theta \\ &\leq \beta M_\alpha t^{-\alpha\beta}\|x\|\int_0^\infty \theta^{1-\alpha}\Phi_\beta(\theta)\, d\theta \\ &\leq \frac{\beta M_\alpha t^{-\beta\alpha}\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\|x\|, \ \forall x \in \mathbb{X} \end{aligned}$$

and

$$\begin{aligned} \|\mathbb{P}_\beta(t)x\|_\alpha &= \left\| \int_0^\infty \beta\theta\Phi_\beta(\theta)R(\theta t^\beta)x\, d\theta \right\|_\alpha \\ &\leq \int_0^\infty \beta\theta\Phi_\beta(\theta)\left\|A^\alpha R(\theta t^\beta)x\right\| d\theta \\ &\leq M\|x\|_\alpha \textstyle\int_0^\infty \beta\theta\Phi(\theta)d\theta \\ &= M\|x\|_\alpha\frac{\beta}{\Gamma(1+\beta)}, \ \forall x \in \mathbb{X}_\alpha. \end{aligned}$$

□

Definition 2.6. ([5, 6]) Let $\mathbb{S}_\beta$ and $\mathbb{P}_\beta$ be operators defined respectively by (6) and (7). Then a continuous function $x : I \to \mathbb{X}$ satisfying for any $t \in [0,T]$ the equation

$$(8)\qquad \begin{aligned} x(t) &= \mathbb{S}_\beta(t)(x_0 - g(x)) \\ &+ \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[(f(s,x(s)) - \int_0^t a(t-s)h(s,x(s))\right] ds, \end{aligned}$$

is called a mild solution of the equation (1).

In the sequel, we set

$$(9)\qquad Kx(t) := \int_0^t a(t-s)h(s,x(s))\,ds.$$

We set $\alpha \in (0,1)$ and we will denote by $\mathcal{C}_\alpha$, the Banach space $C([0,T],\mathbb{X}_\alpha)$ endowed with the supnorm given by

$$\|x\|_\infty := \sup_{t\in I}\|x\|_\alpha, \quad \text{for } x \in \mathcal{C}.$$

3. Main Results

In addition to the previous assumptions, we assume that the following hold.

(H$_1$) The function $f : I \times \mathbb{X}_\alpha \to \mathbb{X}$ is continuous and satisfies the following condition: there exists a function $\mu_1(t) \in L^\infty(I,\mathbb{R}^+)$ such that

$$\|f(t,x,) - f(t,y)\| \le \mu_1(t)\|x-y\|_\alpha$$

for all $t \in I,\ x,\, y \in \mathbb{X}_\alpha$.

(H$_2$) The function $h : \quad I \times \mathbb{X}_\alpha \to \mathbb{X}$ is continuous and satisfies the following condition: there exists a function $\mu_2(t) \in L^\infty(I,\mathbb{R}^+)$ such that

$$\|h(t,x,) - h(t,y)\| \le \mu_2(t)\|x-y\|_\alpha$$

for all $t \in I,\ x,\, y \in \mathbb{X}_\alpha$.

(H$_3$) The function $g : \mathcal{C}_\alpha \to \mathbb{X}_\alpha$ is continuous and there exists a constant b such that

$$\|g(x) - g(y)\|_\alpha \le b\|x-y\|_\infty$$

for all $x,\, y \in \mathcal{C}_\alpha$.

Theorem 3.1. *Suppose assumptions* (H$_1$)-(H$_3$) *hold and that* $\Omega_{\alpha,\beta,T} < 1$ *where*

$$\Omega_{\alpha,\beta,T} = \left[Mb + \frac{\beta M_\alpha \Gamma(2-\alpha)T^{\beta(1-\alpha)}}{\Gamma(1+\beta(1-\alpha))(\beta(1-\alpha))}\Big(\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)} + a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}\Big)\right].$$

If $x_0 \in \mathbb{X}_\alpha$, then (1) has a unique mild solution $x \in \mathcal{C}_\alpha$.

Proof. Define the nonlinear integral operator $F : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ by

$$\begin{aligned}(Fx)(t) &= \mathbb{S}_\beta(t)\,(x_0 - g(x))\,,\\ &+ \int_0^t (t-s)^{\beta-1}\,\mathbb{P}_\beta(t-s)\,[f(s,x(s)) + Kx(s)]\,ds.\end{aligned}$$

where K is given by (9).

In view of Lemma 2.4- (ii), the integral operator F is well defined.

Now take $t \in I$ and $x, y \in \mathcal{C}_\alpha$. We have

$$\begin{aligned}\|(Fx)(t)-(Fy)(t)\|_\alpha &\le \|\mathbb{S}_\beta(t)\,(g(x)-g(y))\,\|_\alpha\\ &+ \int_0^t (t-s)^{\beta-1}\,\|\mathbb{P}_\beta(t-s)\,(f(s,x(s))-f(s,y(s)))\|_\alpha\,ds\\ &+ \int_0^t (t-s)^{\beta-1}\,\|\mathbb{P}_\beta(t-s)\,(K\,x(s)-K\,y(s))\|_\alpha\,ds\end{aligned}$$

which according to Lemma 2.5 and (H_3) gives

$$\begin{aligned}\|(Fx)(t)-(Fy)(t)\|_\alpha &\le Mb\|x-y\|_\infty\\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1}\,\|(f(s,x(s))-f(s,y(s)))\|\,ds\\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1}\,\|(K\,x(s)-K\,y(s))\|\,ds\end{aligned}$$

Since (H_2) and (2) hold, we can write

$$\begin{aligned}\|Kx(s)-Ky(s)\| &= \int_0^s a(s-\tau)\,\|h(\tau,x(\tau))-h(\tau,y(\tau))\|\,d\tau\\ &\le \int_0^s a(s-\tau)\mu_2(\tau)\,\|x(\tau)-y(\tau)\|\,d\tau\\ &\le a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}\|x-y\|_\infty.\end{aligned}$$

Thus, using (H_1) we obtain

$$\begin{aligned}\|(Fx)(t)-(Fy)(t)\|_\alpha \le& Mb\|x-y\|_\infty\\ &+\frac{\beta M_\alpha \Gamma(2-\alpha)\|x-y\|_\infty}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1}\mu_1(s)\,ds\\ &+\frac{\beta M_\alpha \Gamma(2-\alpha)T^{\beta(1-\alpha)}}{\Gamma(1+\beta(1-\alpha))(\beta(1-\alpha)}a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}\|x-y\|_\infty\\ \le& Mb\|x-y\|_\infty\\ &+\frac{\beta M_\alpha \Gamma(2-\alpha)T^{\beta(1-\alpha)}}{\Gamma(1+\beta(1-\alpha))(\beta(1-\alpha))}\|x-y\|_\infty\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}\\ &+\frac{\beta M_\alpha \Gamma(2-\alpha)T^{\beta(1-\alpha)}a_T}{\Gamma(1+\beta(1-\alpha))(\beta(1-\alpha))}\|x-y\|_\infty\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}\\ \le& \Omega_{\alpha,\beta,T}\|x-y\|_\infty\end{aligned}$$

So we get

$$\|(Fx)(t)-(Fy)(t)\|_\infty \quad \le \quad \Omega_{\alpha,\beta,T}(t)\|x-y\|_\infty.$$

Since $\Omega_{\alpha,\beta,T}<1$, the contraction mapping principle enables us to say that, F has a unique fixed point in $\mathcal{C}_\alpha$,

$$x(t)=\mathbb{S}_\beta(t)\,(x_0-g(x))+\int_0^t (t-s)^{\beta-1}\,\mathbb{P}_\beta(t-s)\,[f(s,x(s))+K\,x(s)]\;ds$$

which is the mild solution of (1). □

Now we assume that

(H$_4$) The function $f : I\times\mathbb{X}_\alpha\to\mathbb{X}$ is continuous and satisfies the following condition: there exists a positive function $\mu_1\in L^\infty(I,\mathbb{R}^+)$ such that

$$\|f(t,x)\|\le\mu_1(t),$$

(H$_5$) The function $h : I\times\mathbb{X}_\alpha\to\mathbb{X}$ is continuous and satisfies the following condition: there exists a positive function $\mu_2\in L^\infty(I,\mathbb{R}^+)$ such that

$$\|h(t,x)\|\le\mu_2(t),$$

(H$_6$) The function $g\in C(\mathcal{C}_\alpha,\mathbb{X}_\alpha)$ is completely continuous and there exist $\lambda,\gamma>0$ such that

$$\|g(x)\|_\alpha\le\lambda\|x\|_\infty+\gamma.$$

Theorem 3.2. *Suppose that assumptions* (H$_4$)-(H$_6$) *hold. If* $x_0\in\mathbb{X}_\alpha$ *and*

$$M\lambda<\frac{1}{2} \tag{10}$$

then (1.1) has a mild solution on $[0,T]$.

Proof. Define the integral operator $F:\mathcal{C}_\alpha\to\mathcal{C}_\alpha$ by

$$\begin{aligned}(Fx)(t) &= \mathbb{S}_\beta(t)\,(x_0-g(x))\,,\\ &+ \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\,[f(s,x(s))+Kx(s)]\;ds,\end{aligned}$$

and choose r such that

$$\begin{aligned} r &\ge 2\frac{T^{\beta(1-\alpha)}\beta M_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))(\beta(1-\alpha))}\Big(\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}+a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}\Big)\\ &+ 2M(\|x_0\|_\alpha+\gamma).\end{aligned}$$

Let $B_r=\{x\in\mathcal{C}_\alpha:\ \|x\|_\infty\le r\}$. We proceed in three main steps.

Step 1. We show that $F(B_r) \subset B_r$. For that, let $x \in B_r$. Then for $t \in I$, we have

$$\begin{aligned}
\|(Fx)(t)\|_\alpha &\leq \|\mathbb{S}_\alpha(t)\,(x_0 - g(x))\|_\alpha \\
&+ \int_0^t (t-s)^{\beta-1} \left\|\mathbb{P}_\alpha(t-s) f(s, x(s))\right\|_\alpha ds \\
&+ \int_0^t (t-s)^{\beta-1} \left\|\mathbb{P}_\alpha(t-s) K\, x(s)\right\|_\alpha ds
\end{aligned}$$

which according to (H_4)-(H_6) and Lemma 2.5 gives

$$\begin{aligned}
\|(Fx)(t)\|_\alpha &\leq M\left(\|x_0\|_\alpha + \lambda\|x\|_\infty + \gamma\right) \\
&+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \|f(s,x(s))\|\, ds \\
&+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \|Kx(s)\|\, ds \\
&\leq M\left(\|x_0\|_\alpha + \lambda\|x\|_\infty + \gamma\right) \\
&+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \mu_1(s)\, ds \\
&+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \int_0^s a(s-\tau)\mu_2(\tau) d\tau\, ds.
\end{aligned}$$

Consequently, using the inequality $M\lambda < \frac{1}{2}$, which yields $M\lambda\|x\|_\infty < \frac{r}{2}$ and the choice of r above, we get

$$\begin{aligned}
\|(Fx)(t)\|_\alpha &\leq M\left(\|x_0\|_\alpha + \lambda\|x\|_\infty + \gamma\right) \\
&+ \frac{\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)} T^{\beta(1-\alpha)}}{(\beta(1-\alpha)} \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \\
&+ \frac{\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)} T^{\beta(1-\alpha)}}{(\beta(1-\alpha)} \frac{\beta M_\alpha \Gamma(2-\alpha) a_T}{\Gamma(1+\beta(1-\alpha))}.
\end{aligned}$$

In view of (10) and the choice of r, we obtain

$$\|(Fx)\|_\infty \leq r.$$

Step 2. We prove that F is continuous. For that, let (x_n) be a sequence of B_r such that $x_n \to x$ in B_r. Then

$$\begin{aligned}
f(s, x_n(s)) &\to f(s, x(s)), \qquad n \to \infty, \\
h(t, x_n(s)) &\to h(t, x(s)), \qquad n \to \infty
\end{aligned}$$

as both f and h are jointly continuous on $I \times \mathbb{X}_\alpha$.

Now, for all $t \in I$, we have

$$\begin{aligned}
\|Fx_n - Fx\|_\alpha &\leq \|\mathbb{S}_\beta(t)(g(x_n) - g(x))\|_\alpha \\
&+ \left\|\int_0^t (t-s)^{\beta-1} \mathbb{P}_\beta(t-s)\left(Kx_n(s) - Kx(s)\right) ds\right\|_\alpha \\
&+ \left\|\int_0^t (t-s)^{\beta-1} S(t-s)\left(f(s,x_n(s)) - f(s,x(s))\right) ds\right\|_\alpha,
\end{aligned}$$

which in view of Lemma 2.5 gives

$$\begin{aligned}\|Fx_n - Fx\|_\alpha \leq\ & M\|g(x_n) - g(x)\|_\alpha \\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1}\, \|f(s,x_n(s)) - f(s,x(s))\|\ ds \\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1}\, \|Kx_n(s) - Kx(s)\|\ ds\end{aligned}$$

for all $t \in I$. Therefore, on the one hand using (2), (H_4) and (H_5), we get for each $t \in I$

$$\begin{aligned}\|f(s,x_n(s)) - f(s,x(s))\| &\leq 2\mu_1(s) \text{ for } s \in I, \\ \|Kx_n(s) - Kx(s)\| &\leq \int_0^s a(s-\tau)\|h(\tau,x_n(\tau)) - h(\tau,x(\tau))\|d\tau, \\ &\leq 2\int_0^s a(s-\tau)\mu_2(\tau)d\tau \\ &\leq 2a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)} \text{ for } s \in I;\end{aligned}$$

and on the other hand using the fact that the functions $s \mapsto 2\mu_1(s)(t-s)^{\beta(1-\alpha)-1}$ and $s \mapsto (t-s)^{\beta(1-\alpha)-1}$ are integrable on I, by means of the Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned}&\int_0^t (t-s)^{\beta(1-\alpha)-1}\, \|f(s,x_n(s)) - f(s,x(s))\|\ ds \to 0, \\ &\int_0^t (t-s)^{\beta(1-\alpha)-1}\, \|Kx_n(s) - Kx(s)\|\ ds \to 0.\end{aligned}$$

Hence, since $g(x_n) \to g(x)$ as $n \to \infty$ because g is completely continuous on $\mathcal{C}_\alpha$, it can easily be shown that

$$\lim_{n\to\infty} \|(Fx_n) - (Fx)\|_\infty = 0,$$

as $n \to \infty$.

In other words, F is continuous.

Step 3. We show that F is compact. To this end, we use the Ascoli-Arzela's theorem. For that, we first prove that $\{(Fx)(t) : x \in B_r\}$ is relatively compact in $\mathbb{X}_\alpha$, for all $t \in I$. Obviously,$\{(Fx)(0) : x \in B_r\}$ is compact.

Let $t \in (0,T]$. For each $h \in (0,t)$, $\epsilon > 0$ and $x \in B_r$, we define the operator $F_{h,\epsilon}$ by

$$\begin{aligned}
(F_{h,\epsilon}x)(t) &= \mathbb{S}_\beta(t)\,(x_0 - g(x)) \\
&+ \int_0^{t-h}(t-s)^{\beta-1}\int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta)f(s,x(s))d\theta\,ds \\
&+ \int_0^{t-h}(t-s)^{\beta-1}\int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta)Kx(s)d\theta\,ds \\
&= \mathbb{S}_\beta(t)\,(x_0 - g(x)) \\
&+ R(h^\beta\epsilon)\int_0^{t-h}(t-s)^{\beta-1}\int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta - h^\beta\epsilon)f(s,x(s))d\theta\,ds \\
&+ R(h^\beta\epsilon)\int_0^{t-h}(t-s)^{\beta-1}\int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta - h^\beta\epsilon)Kx(s)d\theta\,ds.
\end{aligned}$$

Then the sets $\{(F_{h,\epsilon}x)(t) : x \in B_r\}$ are relatively compact in $\mathbb{X}_\alpha$ since by Lemma 2.3, the operators $R_\alpha(t),\ t > 0$ are compact on $\mathbb{X}_\alpha$. Moreover, using (H_1) and (4) we have

$$\begin{aligned}
&\|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha \le \\
&\int_0^t (t-s)^{\beta-1}\int_0^\epsilon \beta\theta\Phi_\beta(\theta)\left\|R((t-s)^\beta\theta)f(s,x(s))\right\|_\alpha d\theta\,ds + \\
&\int_{t-h}^t (t-s)^{\beta-1}\int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)\left\|R((t-s)^\beta\theta)f(s,x(s))\right\|_\alpha d\theta\,ds + \\
&\int_0^t (t-s)^{\beta-1}\int_0^\epsilon \beta\theta\Phi_\beta(\theta)\left\|R((t-s)^\beta\theta)Kx(s)\right\|_\alpha d\theta\,ds + \\
&\int_{t-h}^t (t-s)^{\beta-1}\int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)\left\|R((t-s)^\beta\theta)Kx(s)\right\|_\alpha d\theta\,ds.
\end{aligned}$$

Then using (4) and (H_4), we obtain

$$\begin{aligned}
\|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\le \beta M_\alpha\int_0^t (t-s)^{\beta(1-\alpha)-1}\mu_1(s)\int_0^\epsilon \theta^{1-\alpha}\Phi_\beta(\theta)d\theta\,ds \\
&+ \beta M_\alpha\int_{t-h}^t (t-s)^{\beta(1-\alpha)-1}\mu_1(s)\int_\epsilon^\infty \beta\theta^{1-\alpha}\Phi_\beta(\theta)d\theta\,ds \\
&+ \beta M_\alpha\int_0^t (t-s)^{\beta(1-\alpha)-1}\int_0^\epsilon \beta\theta^{1-\alpha}\Phi_\beta(\theta)\left\|Kx(s)\right\|d\theta\,ds \\
&+ \beta M_\alpha\int_{t-h}^t (t-s)^{\beta(1-\alpha)-1}\int_\epsilon^\infty \beta\theta^{1-\alpha}\Phi_\beta(\theta)\left\|Kx(s)\right\|d\theta\,ds.
\end{aligned}$$

Since by (H_5) and (2),

$$\begin{aligned}
\|Kx(s)\| &\le \int_0^s a(s-\tau)\|h(\tau,x(\tau))\|d\tau \\
&\le \int_0^s a(s-\tau)\mu_2(\tau)d\tau \\
&\le a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)},
\end{aligned}$$

using (5c), we deduce for all $\epsilon > 0$ that

$$\begin{aligned}\|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\le \frac{t^{\beta(1-\alpha)}\beta M_\alpha \|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)} \int_0^\epsilon \theta^{1-\alpha}\Phi_\beta(\theta)d\theta \\ &+ \frac{h^{\beta(1-\alpha)}\beta M_\alpha \Gamma(2-\alpha)\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))} \\ &+ \frac{t^{\beta(1-\alpha)}\beta M_\alpha \|\mu_2\|_{L^\infty(I,\mathbb{R}_+)} a_T}{\beta(1-\alpha)} \int_0^\epsilon \theta^{1-\alpha}\Phi_\beta(\theta)d\theta \\ &+ \frac{h^{\beta(1-\alpha)}\beta M_\alpha \Gamma(2-\alpha) a_T \|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))}.\end{aligned}$$

In other words

$$\begin{aligned}\|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\le \frac{h^{\beta(1-\alpha)}\beta M_\alpha \Gamma(2-\alpha)\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))} \\ &+ \frac{h^{\beta(1-\alpha)}\beta M_\alpha \Gamma(2-\alpha) a_T \|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))}.\end{aligned}$$

Therefore, the set $\{(Fx)(t) : x \in B_r\}$ is relatively compact in $\mathbb{X}_\alpha$ for all $t \in (0, T]$ and since it is compact at $t = 0$ we have the relatively compactness in $\mathbb{X}_\alpha$ for all $t \in I$. Now, let us prove that $F(B_r)$ is equicontinuous. By the compactness of the set $g(B_r)$, we can prove that the functions Fx, $x \in B_r$ are equicontinuous a $t = 0$. For $0 < t_2 < t_1 \le T$, we have

$$\begin{aligned}&\|(Fx)(t_1) - (Fx)(t_2)\|_\alpha \le \|(\mathbb{S}_\beta(t_1) - \mathbb{S}_\beta(t_2))(x_0 - g(x))\|_\alpha \\ &+ \left\| \int_0^{t_2} (t_1 - s)^{\beta-1} (\mathbb{P}_\beta(t_1 - s) - \mathbb{P}_\beta(t_2 - s))(f(s, x(s)) + Kx(s))\, ds \right\|_\alpha \\ &+ \left\| \int_0^{t_2} \left((t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}\right) \mathbb{P}_\beta(t_2 - s)(f(s, x(s)) + Kx(s))\, ds \right\|_\alpha \\ &+ \left\| \int_{t_2}^{t_1} (t_1 - s)^{\beta-1} \mathbb{P}_\beta(t_1 - s)(f(s, x(s)) + Kx(s))\, ds \right\|_\alpha \\ &\le I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \|(\mathbb{S}_\beta(t_1) - \mathbb{S}_\beta(t_2))(x_0 - g(x))\|_\alpha \\ I_2 &= \left\| \int_0^{t_2} (t_1 - s)^{\beta-1} (\mathbb{P}_\beta(t_1 - s) - \mathbb{P}_\beta(t_2 - s))(f(s, x(s)) + Kx(s))\, ds \right\|_\alpha \\ I_3 &= \left\| \int_0^{t_2} \left((t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}\right) \mathbb{P}_\beta(t_2 - s)(f(s, x(s)) + Kx(s))\, ds \right\|_\alpha \\ I_4 &= \left\| \int_{t_2}^{t_1} (t_1 - s)^{\beta-1} \mathbb{P}_\beta(t_1 - s)(f(s, x(s)) + Kx(s))\, ds \right\|_\alpha\end{aligned}$$

Actually, I_1, I_2, I_3 and I_4 tend to 0 independently of $x \in B_r$ when $t_2 \to t_1$. Indeed, let $x \in B_r$ and $G = \sup_{x \in \mathcal{C}_\alpha} \|g(x)\|_\alpha$. In view of Lemma 2.5, we have

$$
\begin{aligned}
I_1 &= \|(\mathbb{S}_\beta(t_1) - \mathbb{S}_\beta(t_2))(x_0 - g(x))\|_\alpha \\
&\leq \int_0^\infty \Phi_\beta(\theta) \left\| R(\theta t_1^\beta) - R(\theta t_2^\beta) \right\|_{\mathbb{B}(\mathbb{X})} \|x_0 - g(x)\|_\alpha \, d\theta \\
&\leq \int_0^\infty \Phi_\beta(\theta) \left\| R(\theta t_1^\beta) - R(\theta t_2^\beta) \right\|_{\mathbb{B}(\mathbb{X})} (\|x_0\|_\alpha + G) d\theta
\end{aligned}
$$

from which we deduce that $\lim_{t_2 \to t_1} I_1 = 0$ since by Lemma 2.3 the function $t \mapsto \|R_\alpha(t)\|_\alpha$ is continuous for $t \geq 0$

$$
I_2 \leq \int_0^{t_2} \left\| (t_1 - s)^{\beta - 1} (\mathbb{P}_\beta(t_1 - s) - \mathbb{P}_\beta(t_2 - s)) (f(s, x(s)) + Kx(s)) \right\|_\alpha ds.
$$

Therefore using the continuity of $\mathbb{P}_\beta(t)$ (Lemma 2.4) and the fact that both f and K are bounded we conclude that $\lim_{t_2 \to t_1} I_2 = 0$

$$
\begin{aligned}
I_3 &\leq \int_0^{t_2} \left((t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1} \right) \|\mathbb{P}_\beta(t_2 - s)(f(s, x(s)) + Kx(s))\|_\alpha \, ds \\
&\leq \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1} \right) (t_2 - s)^{-\alpha\beta} \|f(s, x(s))\| \, ds \\
&+ \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1} \right) (t_2 - s)^{-\alpha\beta} \|Kx(s)\| \, ds.
\end{aligned}
$$

Since $-(t_2 - s)^{-\alpha\beta}(t_1 - s)^{\beta - 1} \leq -(t_1 - s)^{\beta(1 - \alpha) - 1}$ because $(t_1 - s)^{-\alpha\beta} \leq (t_2 - s)^{-\alpha\beta}$, we deduce that

$$
\begin{aligned}
I_3 &\leq \frac{\beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta(1 - \alpha) - 1} - (t_1 - s)^{\beta(1 - \alpha) - 1} \right) ds \\
&+ \frac{a_T \beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta(1 - \alpha) - 1} - (t_1 - s)^{\beta(1 - \alpha) - 1} \right) ds \\
&\leq \frac{\beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\beta(1 - \alpha)\Gamma(1 + \beta(1 - \alpha))} (t_1 - t_2)^{\beta(1 - \alpha)} \\
&+ \frac{a_T \beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\beta(1 - \alpha)\Gamma(1 + \beta(1 - \alpha))} (t_1 - t_2)^{\beta(1 - \alpha)}.
\end{aligned}
$$

Hence $\lim_{t_2 \to t_1} I_3 = 0$ since $\beta(1 - \alpha) > 0$.

$$
\begin{aligned}
I_4 &\leq \int_{t_2}^{t_1} (t_1-s)^{\beta-1} \left\|\mathbb{P}_\beta(t_1-s)(f(s,x(s)+Kx(s))\right\|_\alpha ds \\
&\leq \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_{t_2}^{t_1} (t_1-s)^{\beta(1-\alpha)-1} \left\|f(s,x(s))+Bx(s))\right\| ds \\
&\leq \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_{t_2}^{t_1} (t_1-s)^{\beta(1-\alpha)-1}(\mu_1(s)+\int_0^s a(s-\tau)\|h(\tau,x(\tau))\|d\tau)ds \\
&\leq \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}(\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}+a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)})\int_{t_2}^{t_1}(t_1-s)^{\beta(1-\alpha)-1}ds \\
&\leq \frac{(t_1-t_2)^{\beta(1-\alpha)}\beta M_\alpha \Gamma(2-\alpha)}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))}(\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}+a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)})
\end{aligned}
$$

Since $\beta(1-\alpha)>0$, we deduce that $\lim\limits_{t_2\to t_1} I_4 = 0$.

In short, we have shown that $F(B_r)$ is relatively compact, for $t\in I$, $\{Fx : x \in B_r\}$ is a family of equicontinuous functions. Hence by the Arzela-Ascoli Theorem, F is compact. By Schauder fixed point theorem F has a fixed point $x\in B_r$, which obviously is a mild solution to (1). □

4. Example

Let $\mathbb{X} = L^2[0,\pi]$ equipped with its natural norm and inner product defined respectively for all $u,v\in L^2[0,\pi]$ by

$$
\|u\|_{L^2[0,\pi]} = \Big(\int_0^\pi |u(x)|^2dx\Big)^{1/2} \quad \text{and} \quad \langle u,v\rangle = \int_0^\pi u(x)\overline{v(x)}dx.
$$

Consider the following integro-partial differential equation

$$
(E)\quad \begin{cases}
\dfrac{\partial^\beta u}{\partial t^\beta}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \dfrac{\cos(tx)}{1+u^2(t,x)} + \displaystyle\int_0^t e^{-|t-s|}\cos(u(s,x))\,ds, \\[2ex]
u(t,0)=u(t,\pi)=0, \qquad t\in[0,1] \\[2ex]
u(0,x)+\delta_0 \displaystyle\sum_{k=0}^N \int_0^\pi \cos(x-y)u(t_k,y)dy = u_0(x), \qquad x\in[0,\pi]
\end{cases}
$$

where $t\in[0,1]$, $x\in[0,\pi]$, $0<t_1<t_2<...<t_N\leq 1$, and $\delta_0>0$.

First of all, note that f,h,a are given by

$$
f(t,u(t,x)) = \frac{\cos(tx)}{1+u^2(t,x)}, \quad a(t)=e^{-|t|}, \quad \text{and} \quad h(t,u(t,x)) = \cos(u(s,x)),
$$

and hence in (H_4) and (H_5) we take $\mu_1(t)=\mu_2(t)=\pi$. Moreover, $a_1 = \displaystyle\int_0^1 e^{-|t|}dt = 1-e^{-1}$.

Let A be the operator given by $Au = -u''$ with domain

$$D(A) := \{u \in L^2([0,\pi]) : u'' \in L^2([0,\pi]),\ u(0) = u(\pi) = 0\}.$$

It is well known that A has a discrete spectrum with eigenvalues of the form $n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by

$$z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition to the above, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2[0,\pi]$;

(b) The operator $-A$ is the infinitesimal generator of an analytic semigroup $R(t)$ which is compact for $t > 0$. The semigroup $R(t)$ is defined for $u \in L^2[0,\pi]$ by

$$R(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n.$$

(c) The operator A can be rewritten as

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n$$

for every $u \in D(A)$.

Moreover, it is possible to define fractional powers of A. In particular,

(d) For $u \in L^2[0,\pi]$ and $\alpha \in (0,1)$,

$$A^{-\alpha}u = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \langle u, z_n \rangle z_n;$$

(e) The operator $A^\alpha : D(A^\alpha) \subseteq L^2[0,\pi] \mapsto L^2[0,\pi]$ given by

$$A^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n, \quad \forall u \in D(A^\alpha),$$

where $D(A^\alpha) = \Big\{u \in L^2[0,\pi] : \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n \in L^2[0,\pi]\Big\}$.

Clearly for all $t \geq 0$ and $0 \neq u \in L^2[0,\pi]$,

$$\begin{aligned} |R(t)u| &= |\sum_{n=1}^{\infty} e^{-n^2 t}\langle u, z_n\rangle z_n| \\ &\leq \sum_{n=1}^{\infty} e^{-t}|\langle u, z_n\rangle z_n| \\ &= e^{-t}\sum_{n=1}^{\infty} |\langle u, z_n\rangle z_n| \\ &\leq e^{-t}|u| \end{aligned}$$

and hence $\|R(t)\|_{B(L^2[0,\pi])} \leq 1$ for all $t \geq 0$. Here we take $M = 1$.

Set

$$g(u)(\xi) := \delta_0 \sum_{k=0}^{N} \int_0^{\pi} \cos(\xi - y)u(t_k, y)dy.$$

Suppose $\alpha \in (0, \frac{1}{2})$ and

$$\delta_0 < \frac{\sqrt{6}}{2\pi^2 N}. \tag{11}$$

Now

$$\begin{aligned} \|A^{\alpha} g(u)(\xi)\|^2_{L^2[0,\pi]} &= \sum_{n\geq 1} n^{4\alpha}\|z_n\|^2_{L^2[0,\pi]}\,|\langle g(u)(\xi), z_n\rangle|^2 \\ &\leq \sum_{n\geq 1} n^2|\langle g(u)(\xi), z_n\rangle|^2 \\ &= \frac{2}{\pi}\sum_{n\geq 1}|\int_0^{\pi} g(u)(\xi)\, n\sin(n\xi)d\xi|^2 \\ &= \sum_{n\geq 1}\frac{1}{n^2}|\int_0^{\pi}\frac{\partial^2}{\partial\xi^2} g(u)(\xi)\, z_n(\xi)d\xi|^2 \\ &\leq \frac{\pi^2}{6}\|\frac{\partial^2}{\partial\xi^2} g(u)(\xi)\|^2_{L^2[0,\pi]} \\ &\leq \frac{\pi^2}{6}\|g(u)(\xi)\|^2_{L^2[0,\pi]} \\ &\leq \delta_0^2\frac{\pi^2}{6}N^2\pi^2\|u\|^2_{\infty} \end{aligned}$$

and hence $\|g(u)\|_\alpha \leq \lambda\|u\|_\infty + \mu$ where $\lambda = \dfrac{\delta_0\pi^2 N}{\sqrt{6}}$ and $\mu = 0$. Therefore, the condition $M\lambda < \frac{1}{2}$ holds under assumption (11).

Using Theorem 3.2 and inequality Eq. (11) it follows that the system (E) at least one mild solution.

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